A Discussion of Relativistic Phenomena and Construction of Spacetime Diagrams

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1 Introduction

The subconscious brain often makes assumptions about the world around it in order to assess a situation quickly and take action. In many situations it does an excellent job, allowing an athlete to catch a ball while running or a savvy driver to navigate heavy traffic with ease.[1] But assumptions are typically only valid within a certain domain and applying assumptions outside their domain of validity can lead to seemingly paradoxical situations.

When driving on a four-lane road, the scenery is dynamic. Trees and telephone poles sweep by you. Oncoming cars fly towards you. But the car just one lane over, driving in the same direction as you, hardly appears to be moving at all. Thanks to experience, your brain is not confused by any of this. The trees, telephone poles and other surroundings are really stationary. The cars next to you, including the oncoming traffic, are all moving at roughly the same speed as you are, but not all in the same direction. This is so familiar that it hardly warrants mentioning, and while driving you rarely notice it at all.[2]

When you pull into a parking lot, take the keys out of the ignition and start taking off your seatbelt, you experience a moment of panic. Is the car moving forward? Within seconds you realize that your car isn't moving, but the car next to you is reversing out of its spot. (Take a moment to try and remember this feeling.) The fleeting thought that perhaps you forgot to apply the parking brake is caused by the unconscious assumption that your surroundings really are stationary, and any apparent motion of the objects around you is a reflection of your own movement. In most driving situations, this is a perfectly valid assumption (which is why your brain is in the habit of making it) but as with many assumptions, it is not valid in all circumstances.

Suppose two persons, Alice and Bob, are trying to figure out who is taller. This is made difficult because they are standing on opposite banks of a river. If they could stand right next to each other, they would find that they are both the same height. Instead, they have to translate their observations into an interpretation of reality. Since far away objects look smaller than nearby objects, to Alice, Bob looks shorter than herself. To Bob, Alice looks shorter. Both Alice and Bob conclude that they are taller than the other, and based solely on *apparent* height, both are correct, even though the conclusions appear to contradict each other. The problem is that apparent height is not an intrinsic property, but depends on the *actual* height as well as the distance from the observer. Had Alice and Bob taken this into account, they would have concluded correctly that both are the same height.

The effect of distance on apparent size is so familiar that the brain automatically takes it into account when trying to estimate actual size, so that you have to hold up your thumb to confirm a full moon on

[1] Both of these examples showcase the brain's ability to predict where a moving object will be in the future. The author has always been fascinated with how well the brain does this.

[2] The subconscious is also really good at deciding what is important and ignoring everything else. Unfortunately the conscious brain has trouble overriding the subconscious in many situations, which makes sitting through biology class or reading papers on relativity more difficult than it should be.

the horizon has the same apparent size as a full moon high in the sky.[3] Less familiar (largely unknown, actually), is the effect of relative motion on apparent size. Relative motion affects apparent size much less than does distance in typical Earthly settings. In fact, only recently have instruments (fancy rulers) been able to measure the effect at all on a terrestrial scale. But when Alice and Bob are moving very quickly toward or away from one another, they both appear much smaller to each other than distance alone would explain. This phenomenon is called *length contraction*.

Perception is an imperfect method of learning about reality. The fundamental nature of an object should not depend on the observer. Alice and Bob both have particular heights that do not depend on the width of the river because height is an intrinsic property of persons and other objects. Perceiving an object in multiple ways can improve our understanding of the object. Alice can move towards or away from Bob and note the change in his apparent height, convincing herself that apparent height cannot be one of his fundamental attributes. Changing the way we observe something may affect our perceptions of it, but not its physical nature.[4] The apparent size of a far away object is not a property of the object itself, but rather of the relationship between the object and observer, i.e. the distance between them. When that relationship changes, the properties of the relationship, including apparent height, change accordingly. The fundamental nature of an object may be defined as what stays the same no matter how the method of observation changes.

To illustrate the limitations of perception, imagine looking at a mountain far off in the distance. As we walk a few steps towards or away from the mountain, it does not appear to change size.[5] We might conclude that the apparent size of the mountain *is* an intrinsic property, because as we move around, it does not change. Walking towards it, mile after mile, seems to confirm this at first. Only when we get much closer to the mountain does its apparent size change noticeably. The farther away from an object we are, the less changing distance affects the apparent size of an object. At some point, the change in apparent size becomes unnoticeable to the casual observer, but is still measurable with more sensitive methods.

The order of events was once thought to be a fundamental property of those events. Alice wakes up, then makes some coffee, then showers, and leaves for work. Formally, (and we must be very precise in how we phrase things in order to pinpoint the subconscious assumptions we make about the world around us) we might assume that given two events (e.g. making coffee and taking a shower), exactly one of the following statements is true.

1. One event happened before the other.
2. One event happened after the other.
3. The two events happened at the same time (simultaneously).

Determining which is the case might be difficult in practice (for example if two events happen very nearly at the same time) because there are limits to human perception, but our everyday experiences tell us that events have a natural ordering on which everyone would agree. If this were true it would be possible, at least in principle, to make a list of everything that had ever happened, everywhere, and sort it chronologically into an absolute history of the Universe. To an extent, this *is* possible. The Lincoln assassination happened *before* World War II, and all sufficiently long-lived observers would agree. But for some events, the order in which they happen depends on the motion of the observer. The history of the Universe is not as absolute as humans once believed, but depends on the historian. Consequently, an absolute ordering of events cannot be

[3] A well-known optical illusion is that the Moon appears larger when it is on the horizon near some trees or buildings, than when it stands alone in the sky. No matter where the Moon is, it has the same actual size and practically the same apparent size, but the brain's *estimate* of the actual size is larger when the Moon is next to a familiar object like a tree. The brain knows the tree is larger than it appears, so in that case, the Moon must also be larger than it appears. When there are no familiar objects with which to compare, the brain does not artificially increase its estimate of the actual size of the Moon.

[4] In order to observe anything, we must interact with it. Our interactions may change the object, so that measuring the hardness of a teacup with a hammer shatters the teacup, but the method of observation has nothing to do with the essence of what is being observed. This is very important in quantum mechanics which deals with fragile systems unavoidably and unpredictably changed by observation. It is less important here, and we may assume that the act of observing results in no meaningful change in what is being observed, unlike the teacup.

[5] On a clear day in Los Angeles, the mountains to the East are visible and suit this experiment nicely.

a fundamental property of nature, but rather a property of the relationship between the events themselves and the observer. Thus, there is actually a fourth option, in addition to those listed above.

4. The order of the two events depends on the observer.

Given any two events, one and only one of the above statements is true. Some events occur before others, absolutely, in the sense that all observers, regardless of location or viewing angle or motion would agree as to which happened first. Other groups of events cannot be so ordered, so that Alice may observe a runner take off before the starting pistol fires, while Bob observes the pistol firing before the runner starts, and *both Alice and Bob are correct*, even though their statements seem to contradict each other.

This paper was written in response to a friend's inquiry about an animated GIF illustrating time dilation,[6] which is a name for the observation that the time between two events depends on the relative motion between the events themselves and the observer. We discuss how to draw spacetime diagrams, how simultaneity depends on reference frame, and the phenomena of time dilation and length contraction. The goal was to use as little math as possible while still arriving at precise results. Knowledge of algebra and geometry at a high school level should be sufficient to understand the topics presented here.[7]

2 Galilean Relativity

One of the central themes of physics is change; in fact much of physics is devoted to quantifying and categorizing different types of change. Yet some of the most interesting and beautiful ideas in physics describe *invariance*, those properties which stay the same under varying conditions. For instance, we observe that even as the Earth rotates about the Sun traveling millions of miles every year, apples fall to the ground at the same rate, water boils at the same temperature, and very small rocks still float. We conclude that at least within our range of experience, location does not directly affect physical phenomena.[8] At best, we can determine our location *relative* to something else. Measuring atmospheric pressure can tell us our altitude—that is, our location relative to the center of the Earth. Measuring the length of a year can tell us how far the Earth is from the Sun. Charting the positions of the stars and galaxies can tell us our position relative to the stars and galaxies, but there is no way to determine our *absolute* location, nor is there any indication that such a concept is even meaningful. Every point in the Universe is indistinguishable from every other in the sense that the laws of physics are the same everywhere.

What do we mean by a law of physics? A physical law is a description of a phenomenon that seems to work the same way in a broad range of situations. The example we use repeatedly in this paper is the effect of distance on apparent size. Whether Alice and Bob stand a distance apart in a desert, in a city, on Mars, or on a spaceship, distance affects apparent size exactly the same way, so the relationship between distance and apparent size may be considered a physical law. Yet this law does not hold in all circumstances. If a magnifying glass were positioned between the two of them, Alice would appear larger to Bob than distance alone would predict. This physical law, like *every* physical law ever proposed, is incomplete. It could be extended by taking into account the nature of magnifying glasses, but this modified law would also be incomplete. Regardless, identifying patterns in nature by suggesting and testing physical laws helps us learn about our Universe.

In our Universe, distance from an object affects its apparent size, but the location of the object does not. One could imagine a universe that had one special point, which we might define to be the center of that universe by virtue of its uniqueness, regardless of its location. In this universe, apparent size depended not on the distance from observer to observed, but on the distance between the subject and this point. The observer could move anywhere, and as long as the object did not move relative to the center of this

[6] `http://upload.wikimedia.org/wikipedia/commons/e/eb/Time_dilation02.gif`

[7] If the reader understands what it means for two lines to be parallel, and has at least a vague recollection of the quadratic formula, this will do.

[8] Water boils at a lower temperature at the top of a mountain than at sea level, but that is because atmospheric pressure is less there, not because of any intrinsic location dependence. Using a hyperbaric chamber where atmospheric conditions can be controlled, we find the boiling point of water depends on pressure and humidity, but not on physical location.

hypothetical universe, apparent size would remain the same. One could infer the location, or at least the direction, of the center of the universe by noting how apparent size changed as the subject of observation moved around. In our Universe, no such changes have ever been noticed; to our knowledge, there are no "special points" and hence no point that may be called center.

Physical phenomena likewise do not depend on velocity. An apple falls in the same direction whether dropped while sitting in your kitchen or on a train moving at a uniform speed on smooth tracks. The Earth rotates about its axis and revolves about the Sun, which in turn moves throughout our Galaxy. The motion of the apple does not depend on the speed of the train or the Earth or the Sun. We conclude there is no natural "rest frame" for the Universe; there is no way of determining in an absolute sense whether or not we are moving. At best, we can determine our speed *relative* to something else. While we are used to thinking, "That car is moving", such a statement is neither true nor false: it is meaningless. Only statements like, "That car is moving *relative to the road*" are meaningful. This is called the Galilean principle of relativity, which might also be called the relativity of motion. Absolute motion, just as with absolute location, has no observational basis and is not a useful concept.

Acceleration, on the other hand, does have an absolute meaning. On an accelerating train, a dropped apple no longer falls straight down, but rather follows a path curved backwards. The path depends on how fast the train is accelerating, but not on the velocity of the train or its location. Precise measurements of the motion of a thrown baseball or a falling apple determines our acceleration not just relative to the Earth or the Milky Way, but to the Universe itself in an absolute sense. While we cannot unambiguously say whether or not we are moving, or where we are located, we *can* unambiguously say whether or not we are accelerating, and by how much.[9]

3 The Constant Speed of Light

When a no-good-dirty-rotten litterer drops a piece of garbage out of a car window while driving down the freeway, he sees it fly backwards. To someone on the side of the road, it actually continues moving *forward* until it hits the ground. An Olympic javelin thrower will get a running start to throw farther. Experience tells us the speed of a thrown object depends not only on the speed with which it is thrown (zero in the case of the dropped garbage), but also on the speeds of the thrower and observer.

Light behaves in a way that is fundamentally different than massive objects[10] like javelins. In the introductory example, measuring the speed of oncoming traffic yields different results based on our own speed relative to the road. Driving faster makes oncoming traffic appear to be traveling faster. Measuring the speed of the light coming from the headlights yields the exact same result regardless of the speed of the oncoming car or our own speed. If the Olympian were somehow throwing light instead of a javelin, it would make no difference how fast they were running while throwing. This is completely contrary to the way massive objects behave, but has been experimentally confirmed.[11] No one knows why the universe works this way, but to the best of our knowledge, it does.[12]

If light behaved the same way as massive objects, it would single out a particular reference frame as stationary. We could say the speed of light is 670 million miles per hour, and any frame that agrees with this is stationary; frames in which the speed of light is different are considered to be in motion. As it is, *all*

[9] Because a dropped pencil follows a seemingly straight line downwards, we can conclude that the rotation of the Earth, its revolution about the Sun, the Sun's motion about the Milky Way, and the Milky Way's motion throughout the Universe, add up to a very small acceleration. While the Earth's rotation alone results in a speed of over a thousand miles per hour for a person standing at the equator, the acceleration experienced there is only 0.075 miles per hour per second, corresponding to a 0 to 60 time of thirteen minutes. We do not experience the *speed* of the rotation no matter how fast it is; we would experience the *acceleration*, if it were not so small. Yet even this small effect justifies launching rockets near the equator, where the acceleration due to the Earth's rotation slightly offsets gravity.

[10] In this context, massive simply means an object *has* mass, as opposed to massless objects like light itself.

[11] The interested reader can look up the Michelson-Morley experiment.

[12] The simplest questions are often the hardest to answer. Why is the Universe the way it is? Why do massless objects behave differently than massive objects? Why do objects have mass at all? The first step in science is to determine *how* or *what* something is, and later to determine *why* it is that way.

observers measure this speed and would be considered stationary by this criterion.

4 The Relativity of Simultaneity

When we take a photograph, perhaps of two horses crossing a finish line, we may implicitly assume the events depicted occur simultaneously, but this ignores the finite speed of light. The way an object appears in a photograph corresponds to how it appeared *before* the picture was actually taken, and the farther away the object, the greater the difference. When taking a picture of the night sky, we capture not the way the stars are now, but as they were long ago. In this way, a photograph is not really a snapshot in time, but rather captures a continuum of moments corresponding to the distances to the subjects of the photo. Determining the result of a race by photograph gives a slight (and negligible) advantage to the horse closer to the photographer. Nonetheless, a camera with a fast shutter speed is as good a method as any to determine simultaneity, provided we know how far away each object in the photo is. If the light from two stars travels the same distance to the camera[13] and both appear in a photograph, we might conclude that we have captured the appearance of those stars simultaneously, even knowing that the act of capturing was itself not simultaneous to the events captured. This will serve as our *definition* of simultaneous: if two or more events appear in the same photograph (say, a horse crossing the finish line and the jockey raising his arms in victory), and the light communicating those events to the camera travels the same distance as determined by the photographer, then those events occurred simultaneously from the perspective of the photographer. The purpose of this section is to show how different photographers, in relative motion, will take different photographs and therefore draw different conclusions on the simultaneity of events.[14]

The basic idea is that distance depends on relative motion. When flying from LAX to LGA, we could say that we are moving from Los Angeles to Manhattan. Or we could say that Los Angeles is moving away from us and Manhattan is moving toward us. Or we could say that we are moving in an airplane and the Earth is rotating below us. As discussed in the section on Galilean relativity, all of these statements are equivalent. We are as justified saying we are stationary and the Universe moves around us, as we are saying that we are moving through a stationary Universe. The concept of stationary is only meaningful when comparing the motion of two objects; it is a property of the *relationship* between those objects, not of the objects themselves.

We now present a method of describing events that does not depend on how an observer (perhaps in a space ship traveling far from any galaxy, instead of a car) is moving. It is a graphical description called a *spacetime diagram*. The words space and time are combined because we will see that the distinction between them is blurred by motion; that is, when Alice considers two events to happen, say five minutes and ten feet apart, Bob might consider them to take place ten minutes and five feet apart. We need to be able to describe these events in a way that does not depend on a particular definition of same-time or same-place. To do so, we must first decide on such definitions for Alice and potentially different definitions for Bob and determine what similarities and differences result.

We specify the time of an event by a label[15] t, which could stand for seconds after a particular star collapses into a black hole or hours after breakfast. Either amounts to a definition of what $t = 0$ means. Two people may or may not agree on when to start counting seconds, or even on what units to use.[16] Distinctions such as this are arbitrary and therefore important only in the sense that the final answer, whatever it is, should not depend on such decisions. Similarly, we may specify the location of an event by three variables, which might be latitude, longitude and altitude relative to the surface of the Earth off the coast of Africa

[13] It doesn't matter where the stars are at any point after the light leaves them, or what happens to them. Many of the stars we see have long ago flickered out, exploded in a supernova, or collapsed into a black hole, but we won't find out about this until long after these events actually happened.

[14] The reason we specified the camera must have a fast shutter time is so that motion blur is not an issue. Ideally we would like a camera that takes a picture instantaneously. Such a camera could never be constructed, but a camera with an arbitrarily fast shutter speed serves the purposes of our discussion.

[15] I might use the word *coordinate* which really just means label.

[16] Fortnights have always been the author's favorite unit of time.

(talk about arbitrary!), or in terms of distance and direction from a moving train. Neither frame is stationary in a cosmic sense, and that is just fine. Events that happen at the same time are labeled with the same time coordinate, regardless of when they were observed. Alice's and Bob's jobs are to determine when events occurred based on their observations.

An event, formally, is anything that can be assigned a particular time and location, even if different observers assign different times and locations. So a bowling ball hitting a table, a star exploding, and clicking the save button are all valid events, but driving to the store is more accurately described as being a collection of events because it doesn't happen all at once. (Leaving home and arriving at the store, on the other hand, are perfectly valid events that, if defining location relative to your car, happen at the same place, i.e. right at your car.) There are many circumstances in which a collection of events[17] will have, for example, the same latitude and longitude (think of a ball thrown straight up and falling straight down), in which case specifying its height at each moment of time is enough to describe the motion. We will concern ourselves primarily with such examples because it allows us to represent everything with two-dimensional pictures (one dimension for time, and the other for space). Consider two events:

(A) An apple hits the ground at some location

(B) A bowling ball hits the ground at some other location

Suppose Alice is standing exactly half-way between these locations. Alice can use a ruler to determine the half-way point; even if the events happen at different times, by taking a picture of the two events and noting where on the ruler the pitchers were standing, Alice can place herself appropriately. Although we did not specify that the apple and bowling ball fall at the same time, suppose Alice observes this to be the case.[18] This situation is illustrated in Figure 1. Events A and B have the same time coordinate $t = 0$, and so are considered simultaneous by Alice (having occurred at the same time). Alice quite selfishly but conveniently defines locations as "distance from Alice", and so her position is specified by $x = 0$ for all time (since she is always zero distance from herself). Thus her position is described by the set of all points having an x-coordinate of 0. This set is the vertical line in the picture and we will call it Alice's time axis, t.[19] Perhaps confusingly, Alice's time axis describes her *position* over time. In the figure on the left, we have drawn the x-axis perpendicular to the t-axis, but this is arbitrary. To show this, we present the axes at an oblique angle on the right. Everything we say will apply equally well to the two figures, but the figure on the left agrees more with intuition since graphs like this are usually shown with perpendicular axes. The point in showing the oblique axes is that, if the line connecting two events is parallel to (or coincident with) the x-axis, then the two events are deemed to have occurred at the same time from Alice's perspective.[20] Similarly, if the line is parallel to the t-axis, the events are deemed to have occurred at the same location. This is the *definition* of same-time and same-place and, by construction, depends on how Alice is moving, since the reference frame is centered on Alice. Since motion is inherently relative, Alice is justified insisting that she really is stationary (and that the rest of the Universe is moving around her), so this definition is perfectly valid.

Now suppose Bob is running by Alice (moving to the right) at some velocity v[21] and suppose he passes by Alice, giving her a high-five right at $t = 0$. Bob's time axis t' describes his position over time and we

[17] It is tempting to call it a *series* of events, but that implies an ordering, and that is specifically what we want to avoid.

[18] This whole example is set up to discuss how different observers, in relative motion, make different observations.

[19] Why bother calling it the time axis then? We could simply call it Alice's position over time, but then we would have to call these space-Alice's-position-over-time diagrams, and that doesn't have the same ring to it as spacetime diagrams. Backwards as it may seem, the spatial axis is used to determine which events occur at the same time, and the temporal axis is used to determine which events occur at the same place.

[20] The line connecting two simultaneous events represents all events simultaneous to those events. If events A and B are simultaneous, and events B and C are simultaneous, then events A and C are simultaneous. This is called an equivalence relation. Suppose the line connecting two simultaneous events, say M and N, were *not* parallel to the x-axis. Then it would intersect the x-axis somewhere, and this point of intersection would be simultaneous both to the events M and N and to events A and B. The only way this could happen is if all four events were simultaneous, in which case the line connecting M and N is coincident with the x-axis.

[21] As discussed in the section on Galilean relativity, statements like, "Bob is moving with velocity v" are inherently meaningless; when we use such statements, we always mean *relative to Alice*.

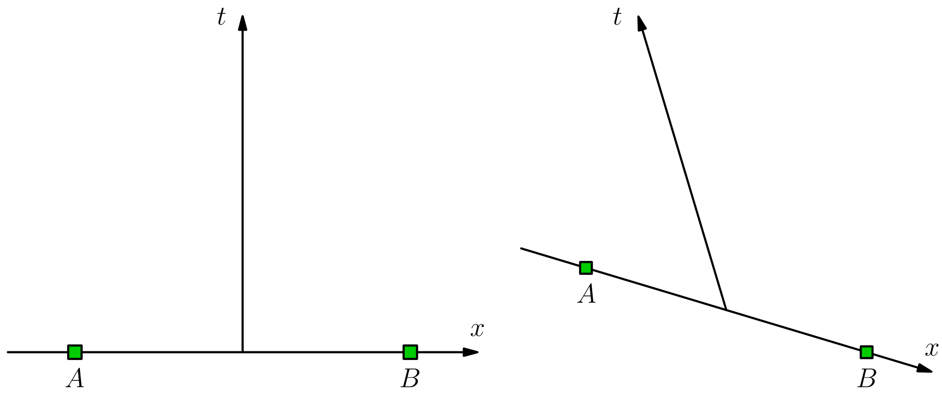


Figure 1: Two simultaneous events, A and B, from Alice's perspective; both have time-label $t = 0$

have lined up everything so that $t' = vt$.[22] This is illustrated in Figure 2. Note that we have chosen the obliqueness of the axes so that Bob's time axis (t') is vertical on the right. This is again arbitrary, but it has the benefit of representing Bob's perspective. We will find that Bob's x-axis (which we call x' to avoid confusion; this is the set of all events that Bob considers simultaneous with $t' = 0$) is horizontal in the figure on the right. There is a symmetric relationship between Alice's observations as depicted on the left, and Bob's on the right. To Alice, Bob appears to be moving to the right, and from Bob's perspective, Alice is moving to the left; in their own frame of reference, both are correct. Bob has his own ruler (or maybe some sort of echo-location device) that he carries with him so that he can keep track of where objects are at any time.

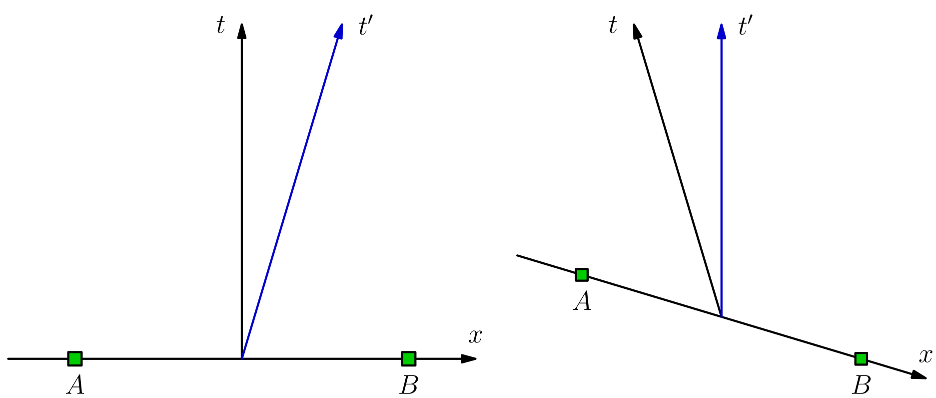


Figure 2: Bob moving to the right, represented by the t' axis

Alice has to translate her observations into an interpretation of reality. To show how she might do this, consider a similar situation where the apple and bowling ball are *thrown* from their locations towards Alice.[23] Suppose both reach Alice at the same time, 1:00. She measures the speed of both the apple and bowling ball

[22] This equation seems innocent enough, but in the interest of being perfectly plain, suppose $v = 60$ miles per hour. Then when $t = 0$, $t' = 0$, and when $t = 1$ hour, $t' = 60$ miles. It may seem odd that t' has units of distance, and it is typical to use the speed of light as a way of converting between hours and miles. Discussing this would be unnecessarily confusing as this section is largely qualitative, but the distinction would be important if we were doing computations.

[23] This example is actually severely flawed since we talk about when and where events occur as if they occur at the same time and place from all perspectives. All of the calculations used in the next few paragraphs are based on the traditional, absolute notions of space and time. We will point out a few implicit, invalid, assumptions as we make them. The only point of this example is to illustrate how light behaves differently than massive objects like apples and bowling balls.

to be 30 miles per hour (this is of course relative to her), and with her ruler she knows that the throwers were both 30 miles from her, but in opposite directions.[24] Alice correctly infers that both apple and bowling ball were thrown, simultaneously, at 12:00. This is shown in Figure 3, with event G representing when the apple and bowling ball reach Alice.

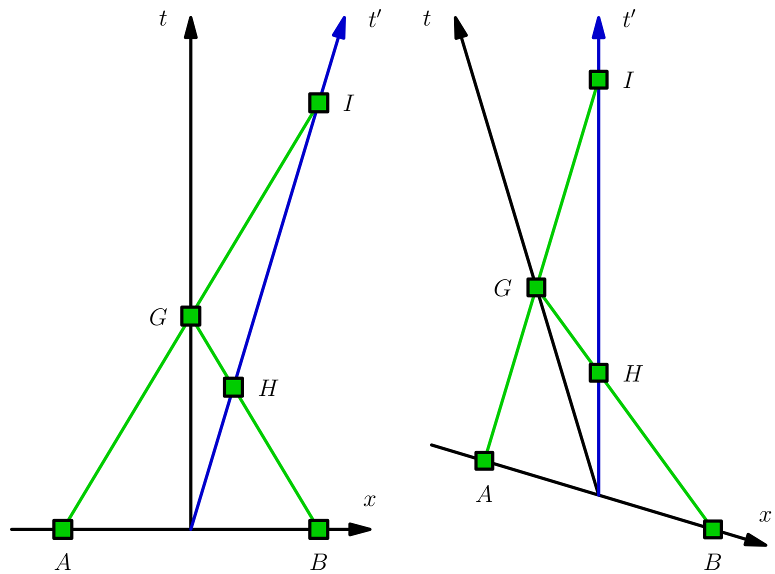


Figure 3: An apple and bowling ball are thrown towards Alice and Bob.

From Alice's perspective, Bob is moving to the right and therefore meets the bowling ball on its way to Alice (but does not interfere with its motion). If Bob is moving 10 miles per hour to Alice's right, then he encounters the bowling ball at 12:45[25] (this is event H; Bob is 7.5 miles from Alice at this point[26]), but it is also traveling 40 miles per hour *relative to Bob*,[27] since the measured speed depends on Bob's motion. The apple, continuing past Alice, reaches Bob at 1:30 (event I; Bob is 15 miles from Alice), but is traveling 20 miles per hour relative to Bob. Bob quickly determines that at 12:00, he was right next to Alice, and the apple and bowling ball were each 30 miles away, being thrown. Bob agrees with Alice that the apple and bowling ball were thrown simultaneously.

Now let's return to the original example. The apple falls and hits the ground 30 miles from where Alice is standing. The bowling ball hits the ground 30 miles from Alice on the other side. Light leaves both the apple and the bowling ball, heading towards Alice, represented by the red lines in Figure 4. Just for this example, let's pretend the speed of light is 30 miles per hour, regardless of who is measuring (in fact, light travels 670 million miles per hour, regardless of who is measuring). We could always redefine the length of a mile, or how long an hour is, to accomplish this. Then if Alice takes a picture (with a wide-angle lens) at 1:00, coinciding with the light reaching Alice, the picture will show the apple hitting the ground on one side, and the bowling ball hitting the ground on the other. Just as before, Alice infers that both the apple

[24] The position of the throwers is only important when they threw; after that they are free to walk off and it doesn't affect Alice's observations. Please ignore the difficulty of throwing a bowling ball thirty miles.

[25] This is according to Alice's clock. We will see in the next section that Alice's clock does not measure time in the same way that Bob's clock does, regardless of what types of clocks Alice and Bob are using. Bob's clock actually reads a time slightly before 12:45.

[26] This is according to Alice's ruler. We will see later that Alice's ruler does not measure distance in the same way as Bob's, regardless of what type of rulers (or echo-location devices) they use. Bob's ruler shows a distance slightly less than 7.5 miles.

[27] This assumes that to determine the speed of the bowling ball relative to Bob we simply add the speed of the bowling ball to the speed of the observer. Prior to the discovery of relativistic phenomena, this seemed the obvious way to determine relative speed and matched measurements of head-on collisions between jousters and well-executed high-fives. Although we do not explain the relativistic correction to this, the actual velocity measured by Bob would be slightly less than 40 miles per hour.

and bowling ball hit the ground, simultaneously, at 12:00. Just as before, Bob encounters the light from the bowling ball on its way towards Alice, sometime before 1:00. The light from the apple reaches Bob sometime after 1:00.[28] When Bob measures the speed of the light from both the apple and bowling ball, he finds that the light from the apple is traveling the same speed as that from the bowling ball, *not* more slowly as may be expected. When he looks at where the light originated, he finds that in both cases the light has travelled the same distance.[29] He therefore concludes the bowling ball must have hit the ground *before* the light from the apple. These events are *not* simultaneous from Bob's perspective.

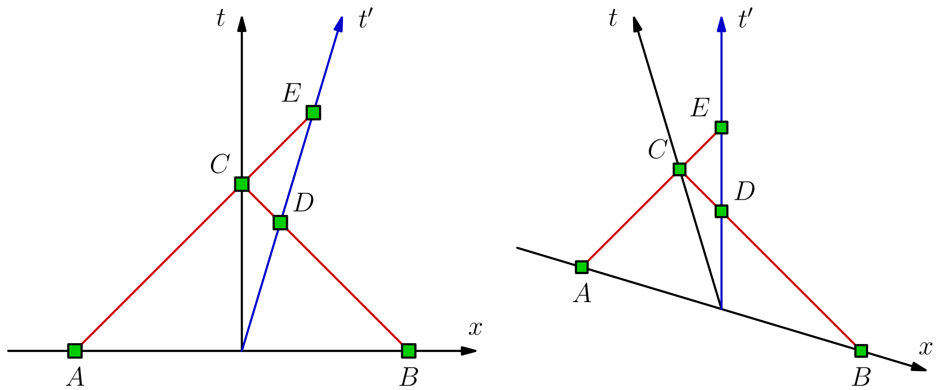


Figure 4: Light rays (in red) reflected by the bowling ball and apple at the moment they hit the ground

If light behaved the same as massive objects, the light from the bowling ball would be traveling faster than the light from the apple, in Bob's frame of reference. Bob could use the difference in speeds to account for the difference in observation times. As it is, the only way to account for *seeing* the apple hit the ground after seeing the bowling ball hit is to infer that the apple really did hit the ground after the bowling ball.

The blue lines that have been added in Figure 5 represent the motion of the tick marks on Bob's ruler where the apple and bowling ball hit. Just as the apple and bowling ball are moving to the left in Bob's frame of reference, the tick marks are moving to the right in Alice's frame of reference and are stationary in Bob's. Although they are moving to the right along with Bob, they stay the same distance apart and the same distance from Bob. Bob considers events that happen along the blue lines to occur at the same place as where the apple and bowling ball hit the ground, respectively. Sometime after the bowling ball hits the ground, a football hits the exact same tick mark. This is event F. The timing is chosen specifically so that light from the football reaches Bob just as the light from the apple does (event E); Bob knows that the light rays from both events are traveling the same speed, and both events happened the same distance away according to his ruler. Since the light reached him at the same time, the apple must have hit the table at the same time as the football—these events are simultaneous from Bob's perspective.

Figure 6 adds a blue line showing that, to Bob, events A and F are simultaneous. This is a line of constant t' and is thus parallel to Bob's x'-axis, shown in Figure 7.[30] Figure 8 shows just Alice's and Bob's

[28]Factoring in the effects of the relative motion between Alice and Bob, the light from the bowling ball actually reaches Bob around 12:42, according to his clock, while the light from the apple reaches him around 1:25, but it isn't at all obvious why this is the case. To explain where these times came from, we need to understand time dilation and length contraction, which will be discussed later. For the purposes of this example, all that matters is that the light from the bowling ball reaches Bob *before* it reaches Alice, while the light from the apple reaches him *afterwards*.

[29]From Bob's perspective, he is stationary, the bowling ball lying on the ground is moving toward him, and the apple is moving away from him. So by the time the light reaches him, the bowling ball is closer to him than the apple. When the light from the apple reaches him, it's even closer. But all the matters in determining simultaneity is how far the *light* has traveled, not the location of the apple or bowling ball, and in both cases, the light has traveled the same distance from Bob's perspective. *Not* 30 miles as Alice determines, but a greater distance due to the effects of length contraction. Since distance is contracted the same amount whether we are moving towards an object or away from it, the distance travelled by the light from the apple is contracted by the same amount as the distance travelled by the light from the bowling ball.

[30]Recall that Bob's x'-axis is *defined* to be the set of all events simultaneous to $t = t' = 0$ and is thus the line passing through

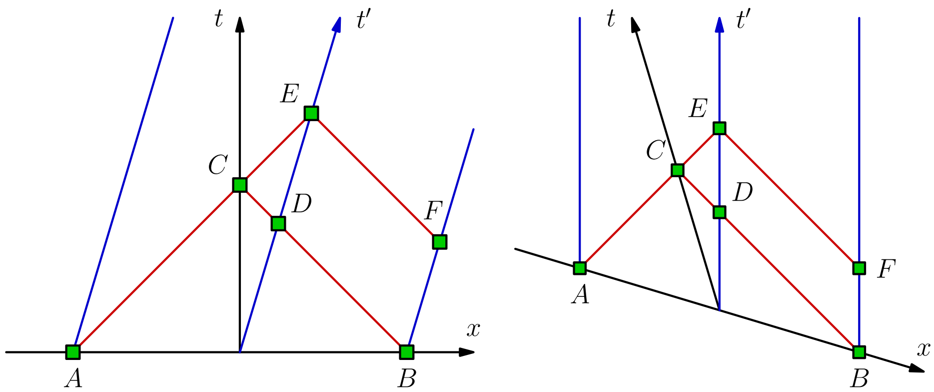


Figure 5: Another light ray leaving a football as it hits the same tick mark on Bob's ruler as did the bowling ball

axes, without events or light rays. It is not obvious, but the angle, θ, Bob's t'-axis makes with Alice's t-axis is equal to the angle Bob's x'-axis makes with Alice's x-axis. This is a consequence of requiring that light travel the same speed in all frames. It can be verified by actually calculating the slopes of all the lines in the figures.

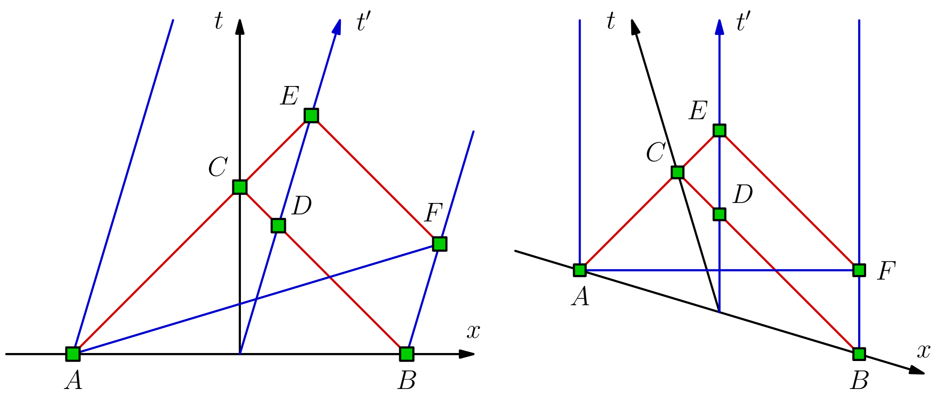


Figure 6: Bob deems events A and F to be simultaneous.

Figure 7 is especially revealing (though cluttered) because it illustrates the chronology from both Alice's and Bob's perspective. Alice's observations lead her to conclude the apple and bowling ball hit the ground just as Bob passes by.[31] Bob's observations lead him to conclude that first the bowling ball hit the ground, then he ran past Alice, and finally the apple hit the ground.

Bob did not take into account the relative motion between Alice and himself. If he conceded that Alice were stationary and he were moving, he could take his motion into account and reach the same conclusion of simultaneity as Alice. Thus Bob can always predict what Alice's observations will be by considering the difference in reference frame. But Bob is just as entitled as Alice to claim that he is stationary. Because Alice and Bob cannot agree who is actually moving (in an absolute sense), they cannot agree on an absolute ordering of events. By considering their relative motion, Bob can put himself in Alice's shoes and determine

the intersection of the t- and t'-axes, parallel to the line connecting events A and F.

[31] We again emphasize the difference between *observing* something happen and inferring when it actually happened. The act of observation always occurs *after* the event, provided the event happened a non-zero distance away in the observer's frame of reference.

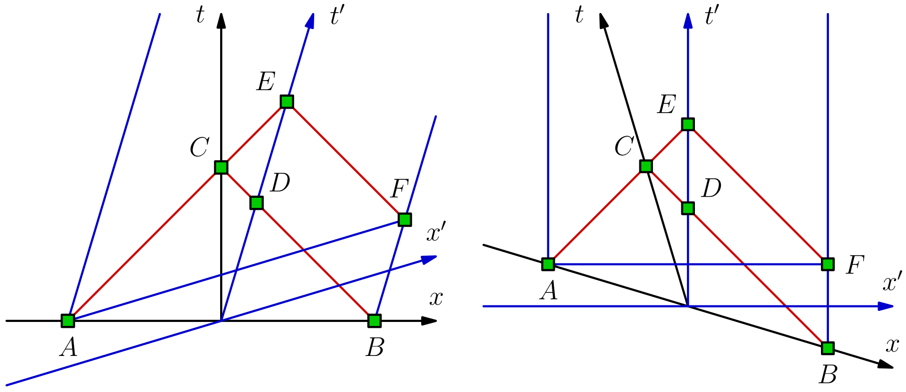


Figure 7: Bob's x-axis added to the figure

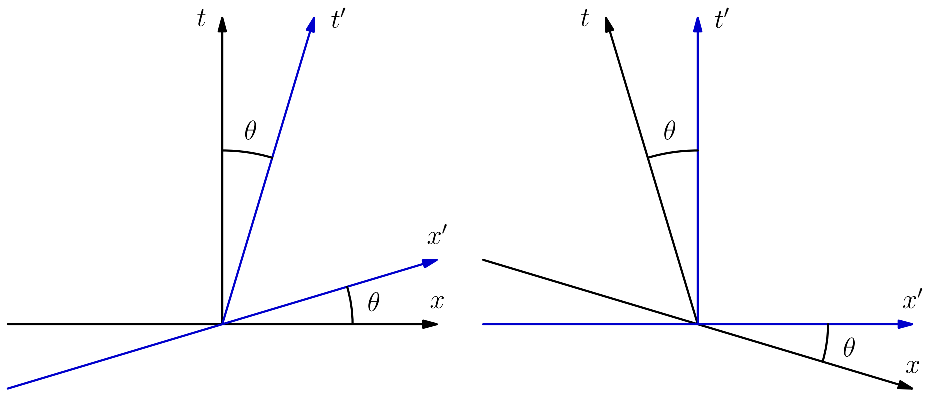


Figure 8: Alice's (black) and Bob's (blue) spacetime axes

how Alice would order the events, and Alice can do the same for Bob, but neither Alice nor Bob can assign a universal order of events without a definition of stationary, a rest frame for the Universe. As the Galilean principle of relativity says, there is no such frame, and no universal ordering of events, no absolute definition of simultaneous.

5 Time Dilation

Let's say Alice wants to play a trick on Bob and gives him a new wristwatch which runs slow by a very small amount: for every 10,000 seconds that passed in reality, the watch will only tick 9,999 times. Watching the second hand sweep buy, Bob would never notice such a small difference, but over time it would start to add up. After a day, the watch would be off by less than nine seconds. After a week, the watch would be one minute slow. Bob would have to re-synchronize his watch every few weeks, a relatively minor inconvenience all things considered (Alice should have put his stapler in jello).

To avoid running late, Bob needs to take into account the inaccuracy of the watch when determining the time between two events. Relative motion makes clocks appear to move more slowly, just like Bob's watch. Two observers in relative motion will experience time differently: they will assign different values to the time between events. Just as there is no natural ordering to events, there is no universal scale against which to measure duration. It is tempting to say that motion affects the passage of time, and this is true in the same way that walking toward a mountain makes it bigger, but in both cases the change is only in the relationship between observer and what is being observed.

Every clock counts intervals of a predetermined duration. A grandfather clock counts swings of a pendulum, with one second defined as a single swing. An atomic clock counts vibrations of a cesium atom, with one second defined as a certain number of vibrations. An hourglass counts falling sand, with one second defined as a certain amount of fallen sand. The clock we will consider defines a second as the time it takes light to travel a certain distance. But the distance between two events depends on the frame of reference. If we define distance in terms of latitude and longitude, then driving across the United States involves traveling a great distance. If instead we define distance relative to our car, then in such a trip we haven't moved at all (but the entire United States has)! The distinction is not physically meaningful,[32] but when combined with the constant speed of light it has interesting results. The speed of light is equal to the distance light travels divided by the time it takes to travel that distance. By simply moving the reference frame (or equivalently, moving yourself relative to the reference frame), we can change the distance light appears to travel. But no matter how the reference frame moves, the speed of light stays the same. Two persons in relative motion will observe the same light travel different distances, and therefore the trip time must be different to maintain a constant speed of light.

Qualitatively, this is a description of time dilation: the rate at which a clock measures time depends on how it is moving relative to an observer. Two identical clocks initially synchronized but then set in motion relative to each other will read different times when later compared. How different depends on how fast they are moving relative to each other and for how long.

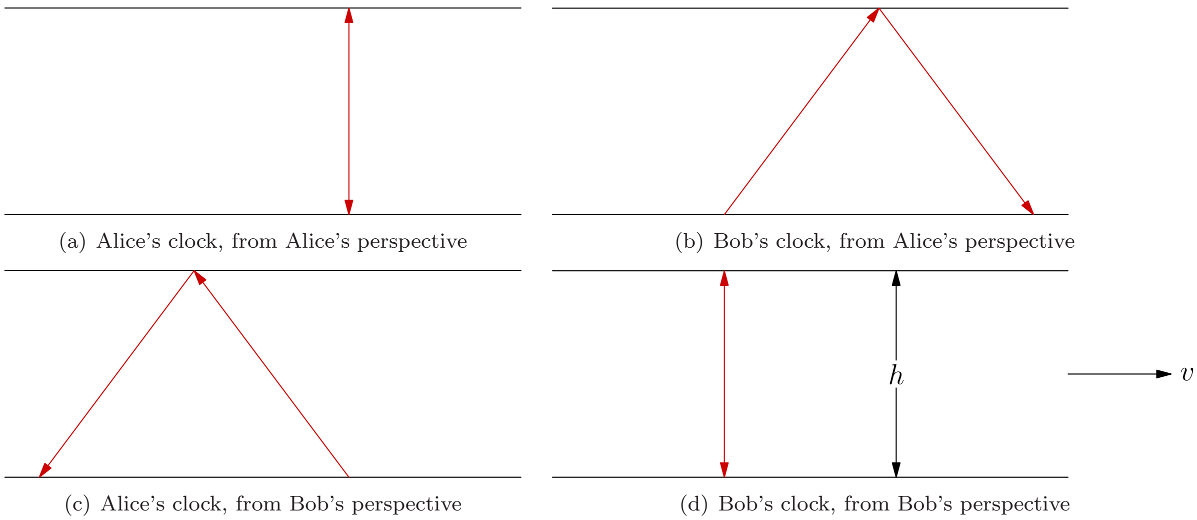
At time $t = 0$, a laser shines light towards a mirror a distance away which reflects the light back to the emitter. There-and-back-again is defined to be one second. Of course, light travels pretty fast, so the mirror needs to be far away—half a light-second[33]—in order for the light to take a second to travel to the mirror and back. We could measure time in minutes or fortnights and everything that follows would be the same.

Suppose Bob takes such a clock on a train passing by a station with velocity v moving to the right, and Alice is standing on the platform holding an identical clock. The clock is held so that the mirror is situated above the laser. From each of their perspectives, the light from their own clock moves up and down, while the light from the other clock appears to move diagonally. From Alice's perspective, by the time Bob's light hits the mirror, the whole train (including the mirror) has moved a distance to the right. The light reflects

[32] A train station leaves a train traveling sixty miles per hour. . .

[33] A light-second is the distance light travels in one second. Similarly, a light-year is the distance light travels in one year, and a light-fortnight is. . . you get the idea.

off the mirror, and by the time it returns to the emitter, the train and the emitter have moved even farther. So the light travels diagonally from Alice's perspective. As Bob moves away from the station, he is also moving away from the light of Alice's clock and so it too appears to move diagonally. This is depicted in Figure 9, which shows Alice's perspective on the top, and Bob's perspective on the bottom, with Alice's clock on the left, and Bob's clock on the right. Note that in these diagrams, there is no time axis, only two spatial dimensions (up-and-down and left-and-right), in contrast to the spacetime diagrams used in the previous section.



(a) Alice's clock, from Alice's perspective

(b) Bob's clock, from Alice's perspective

(c) Alice's clock, from Bob's perspective

(d) Bob's clock, from Bob's perspective

Figure 9: Two identical clocks moving with respect to each other. Alice's perspective is shown on the top row, and Bob's perspective is shown on the bottom. The light from each person's clock moves vertically from their own perspective.

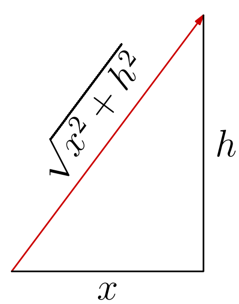


Figure 10: Illustration of the diagonal distance traveled by light in Bob's clock as seen by Alice

Let's focus on Bob's clock, from Alice's perspective (top right in Figure 9). Suppose the train travels a distance $x = v\tau$ in the time it takes for the light to reach the mirror (one half its total journey). From Figure 10 and the Pythagorean theorem, the light travels a distance $\sqrt{x^2 + h^2}$ where h is the total height of the clock: the distance from the laser to the mirror or half a light-second. Since the light travels this distance in the same time τ that the train takes to travel a distance x, we have

$$c\tau = \sqrt{x^2 + h^2} \qquad v\tau = x$$

where c is the speed of light, one light-second per second. Substituting and squaring both sides,

$$c^2\tau^2 = x^2 + h^2 = v^2\tau^2 + h^2$$
$$(c^2 - v^2)\tau^2 = h^2$$
$$\tau^2 = \frac{h^2}{c^2 - v^2} = \frac{h^2/c^2}{1 - \frac{v^2}{c^2}}$$
$$\tau = \frac{h}{c} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\frac{h}{c}$$

where $\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}$ is called the *Lorentz factor.* Since we specified h as half a light-second and c is one light-second per second, h/c is simply half a second (the time it takes light to travel a distance h). Doubling the time to include the return trip, the total amount of time it takes the light to travel from the emitter to the mirror and back is γ seconds from Alice's perspective. For Bob, the time is one second, because we *defined* one second to be the time it takes for the light to make the round trip. If Bob counts off multiple round-trips, whatever amount of time passes for Bob, more time passes for Alice by a factor of γ (which is always greater than one since v^2 is always positive).

It is worth noting how the situation would be different if light behaved intuitively, i.e. if its speed were not constant. Suppose for example that Bob threw a ball straight up at some speed u. Again, from Alice's perspective on the platform, the ball moves diagonally, but with a speed greater than u because the speed from the train adds to the speed of the ball, just as the speed of the running javelin thrower adds to the speed of the thrown javelin. Thus, from Alice's perspective, the ball has a longer (diagonal) distance to travel, but is also moving faster and thus takes the same amount of time. This is different than light, which travels at the same speed from all perspectives.

This is not some illusion affecting only light. Suppose Bob's heart beats once per second, as measured in his own frame of reference, so that every time a pulse of light reaches the emitter, his heart beats. Alice also sees Bob's heart beating in unison with his clock,[34] but because Alice sees his clock moving more slowly, she also sees his heart beating more slowly, sees him breathing more slowly, aging more slowly. Bob's entire notion of the passage of time is synchronized with a clock stationary in his frame of reference; if it weren't, the difference between his biological clock and the mechanical one beside him could be used to determine Bob's speed, the speed of the train, in an absolute sense, which would imply the existence of a stationary frame of reference, one in which his heartbeats and breathing corresponded to the light-based clock. By the Galilean principle of relativity, no such frame exists, and therefore Alice observes Bob aging more slowly the faster he is traveling relative to her.[35]

For the range of relative motion normally experienced on Earth, this difference is barely noticeable by the most sensitive instruments available. If Alice is on the ground and Bob flies overhead in a supersonic jet traveling at 3.5 times the speed of sound (as fast as virtually any manned jet currently in use), circling the Earth over and over for a hundred years, then Alice would see Bob's last heart beat happen just 0.02

[34] There are two events here, light reaching the emitter and Bob's heart beating, and we are claiming that all observers infer these two events are simultaneous, regardless of how the observer is moving. This seems to violate the conclusion of the prior section, that simultaneity is relative. As we mentioned in the Introduction, only *some* events cannot be ordered absolutely; other events may be ordered in a way with which all observers would agree (like the Lincoln assassination occurring before World War II). Recall the dependence on relative motion was a consequence of the difference in transmission time of the light rays used to observe those events (the light from the bowling ball took less time to reach Bob than did the light from the apple, while both light rays took the same amount of time to reach Alice). If one observer sees two events happen at the same place (on the train) at the same time, then the transmission time of light is exactly the same since it's the *same* light and *all* observers draw the same conclusion. It is only when two events are separated by distance or time that it becomes more difficult to order them because different light rays are used in observation.

[35] This wording seems to imply that our understanding of the Universe, embodied in physical laws, *forces* the Universe to be a certain way. It would be more accurate to say something like, if the Galilean principle of relativity were a valid description of the Universe, then Alice would observe Bob aging more slowly. Analogous experiments using subatomic particles instead of persons *do* exhibit this apparent slowing down of time, lending support to the theory of relativity.

seconds before hers.[36] Time dilation then has no practical consequence for terrestrial applications, simply because light travels so fast relative to, for example, the speed of sound. For all practical purposes, Alice and Bob age at the same rate and would never notice any difference.

Figure 11 shows the Lorentz factor as a function of velocity. Note that γ barely changes until velocity exceeds, say, half the speed of light. Past that point however, γ explodes towards infinity as relative velocity approaches the speed of light, so that a linear graph cannot adequately express its behavior. Thus, the figure on the right uses a log scale for the vertical axis[37] to show the exponential nature of the Lorentz factor. As an example, if the train were traveling at 670 million miles per hour relative to Alice, then she would live 23 years, watching Bob age only one year.

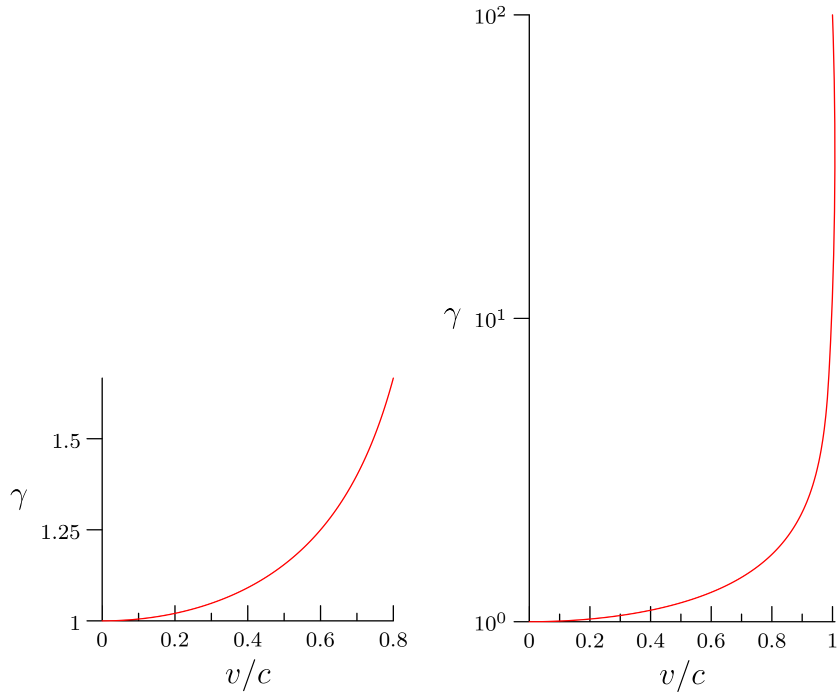


Figure 11: Lorentz factor as a function of velocity, plotted on a linear scale at left and a log scale at right.

In the example of two persons trying to compare height from a distance, both persons thought the other looked smaller, by the same amount, because distance affects size the same for both. A person unfamiliar with the effect of distance on apparent height might go through the following thought process when hearing about this for the first time: Alice looks smaller to Bob because she is far away, so Bob must appear larger to Alice. But of course, both Alice and Bob looks smaller than the other, as paradoxical as this might initially seem. Similarly, relative motion affects the perceptions of both parties equally, making each believe the other is moving in slow motion. In this example, instead of comparing height, Alice and Bob might (quite morbidly) compare time of death. To Alice, Bob is moving in slow motion and, all else being equal, appears to outlive her. To Bob, Alice is moving in slow motion and outlives him. Comparing time of death is subject to the same observational dependence on relative motion that was discussed in the previous section on simultaneity. Deciding who lives longer depends on who is keeping track. This has real implications when, for example, synchronizing a clock on the ground with a clock on a satellite orbiting above the Earth. In fact, the clocks on the GPS satellites are intentionally biased to tick, say 10,000 times for every 9,999 ticks

[36]Again, what we mean is Alice concludes based on her observations that Bob's heart beat at a specific time; Alice *sees* or *hears* it sometime later. English seems to have a built-in assumption that light travels infinitely fast, so that an event happens at the same time that we see it happen, and it is tedious to continue pointing out the difference.

[37]Log scale means that for the vertical axis, each tick mark is ten times the tick mark below it, starting with $10^0 = 1$.

of a clock on the ground, to compensate for the relative motion between them. Knowing that a GPS user on the ground will see the satellite moving in slow motion, the satellites are programmed to transmit signals *faster* than normal to make up for it.

6 Length Contraction

Distance shrinks apparent size in all directions: a distant person looks shorter and thinner than when nearby. Their head maintains the same proportion to their body. One could imagine a universe that worked differently: distance makes a person look thinner, but not shorter. In such a universe, a far-away person would look like a pencil instead of a miniature version of themselves. Length contraction is similar: it does *not* shrink apparent size in all directions, only in the direction of motion. When swimming towards Bob, he looks shorter to Alice the faster she swims. When running towards him, his belly protrudes less. When crab-walking by, his eyes look closer together.[38]

It may not be obvious, but measuring length depends on the notion of simultaneity. To measure the length of a turtle with an adjacent ruler, first we identify the tick mark closest to its beak, then the tick mark closest to its tail. But if the turtle is moving, by the time we look from the beak to the tail, the turtle has moved forward (slightly) leading to a length measurement shorter than reality. Ideally, we would look at both tail and beak *simultaneously*, but even if we couldn't do that, we could take into account the velocity of the turtle and backtrack where the tail *was* when we measured the beak. Regardless, the question is, when the beak was *here*, where was the tail?[39] Because simultaneity is relative, the answer to this question depends on who is asked. Two persons in relative motion will determine different positions for the tail simultaneous to a measurement of the beak, because they have different definitions of simultaneous, and will thus ascribe different lengths to the turtle. Just as there is no natural ordering to events and no universal clock, there is no universal ruler and no absolute way of determining the length of an object. At best, we can take into account our own motion *relative to what is being measured*, and ask, if we were moving right alongside the turtle, at the same speed, how long would we measure it to be? This length is known as the rest length, and is an intrinsic property of an object; different observers may measure different lengths, but all can take their relative motion into account to determine the rest length of the turtle (or whatever), just as they need consider the distance from the turtle in examining its apparent size.

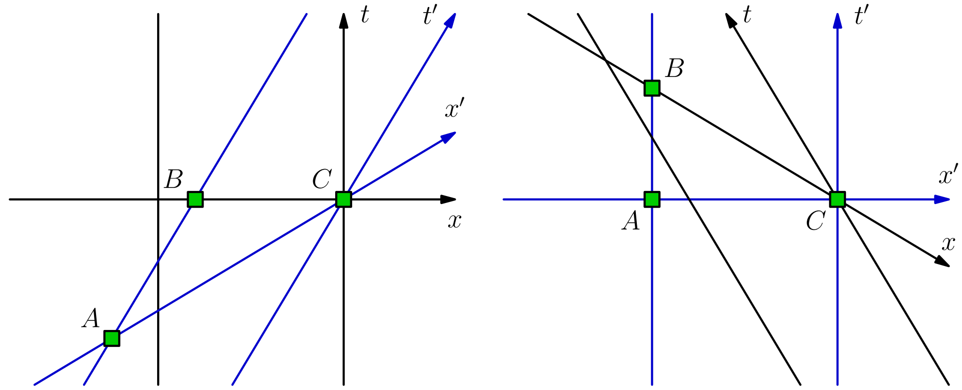
Suppose from Alice's perspective, Timmy the Turtle is moving to the right with velocity v. Alice has a ruler which is stationary from her perspective; at time $t = 0$, Timmy's beak reaches the right side of the ruler. Either by taking a picture, or by some other means, she determines that while she was looking at the beak, Timmy's tail was to the right of the left side of the ruler, so she decides Timmy is shorter than the ruler. What do things look like from Timmy's perspective? Alice is moving to Timmy's left (actually, Alice is behind Timmy, but let's take that as meaning to the left), and Timmy agrees that at $t = t' = 0$, his beak was aligned with the right side of Alice's ruler,[40] which is also moving to the left. The situation is illustrated from both Alice's (on the left) and Timmy's (on the right) perspective.

The construction of Alice's t- and x-axes, and Timmy's t'- and x'-axes is exactly the same as in the section on spacetime diagrams, but all the details have been omitted. The t-axis represents the position of the right side of the ruler over time. The other line, parallel and to the left of the t-axis represents the position of the left side of the ruler. From Alice's perspective, both sides are stationary and thus these lines are vertical, while Timmy sees them move to the left. Timmy's t'-axis is the position of his beak over time; the other line parallel and to the left of the t'-axis is the position of Timmy's tail. Alice's x-axis is the set of

[38] These examples all assume that Bob is oriented the same way as Alice, or possibly facing her so that Bob's left is Alice's left or right and not her up or forward.

[39] Contrast this with measuring the turtle's height. It isn't jumping up and down, so the height of its shell doesn't change as it moves, neither does its girth; the only thing we have trouble measuring is its length, because that is the direction in which it is moving. That's why length contraction only affects size in one dimension, the direction of motion.

[40] We have two events: Timmy's beak is in a specific place at a specific time, and the right edge of the ruler is in a specific place at a specific time. If Alice observes these two events to be at the same time and same place, then so do all observers, including Timmy.



(a) Alice's Perspective (b) Timmy the Turtle's Perspective

Figure 12: Illustration of a turtle being measured

all events simultaneous to the event of Timmy's beak reaching the right side of the ruler, which we call event C. To determine where Timmy's tail was when his beak was at the right side of the ruler, we need only find the intersection of the line representing Timmy's tail with Alice's x-axis, which is event B lying to the right of the left side of her ruler. This is from Alice's perspective. From Timmy's perspective, we need to use the x'-axis instead of the x-axis and find its intersection with the line representing Timmy's tail, event A. As we can see, it is to the left of Alice's ruler, so Timmy believes himself to be longer than the ruler.[41]

Suppose Timmy, using his own ruler, determines his length to be ℓ. From Timmy's perspective, the time it takes the ruler to move from his beak to his tail (recall that in Timmy's frame of reference, it is the ruler that is moving) is simply the distance it travels (ℓ, the distance from beak to tail) divided by the ruler's velocity v. As we have discussed, to Timmy, Alice appears to experience time at a different rate. She is moving in slow motion so that for every γ seconds that Timmy experiences, Alice experiences one second; equivalently, for every one second Timmy experiences, Alice experiences $1/\gamma$. Thus if Timmy determines it will take an amount of time equal to ℓ/v for the ruler to pass from his beak to tail, Alice must observe $\tau = (\ell/v) \cdot (1/\gamma) = \ell/(v\gamma)$. From Alice's perspective, Timmy is moving with velocity v and takes $\tau = \ell/(v\gamma)$ time for his tail to reach the right side of the ruler, thus his length must be $v\tau = \ell/\gamma$. Recalling that γ gets larger with velocity, the faster Timmy is moving, the shorter he appears to Alice.

To Timmy, it is the ruler that is moving, and it is the ruler whose length is contracted by that motion. If by coincidence the ruler itself had the same rest length ℓ as Timmy, Alice sees that Timmy's length is contracted, and he appears to be shorter than the ruler. Timmy sees the *ruler's* length contracted, and he concludes that he is longer than the ruler. Recall the two persons comparing heights from across a river: both see themselves as being taller than the other, even though both are actually the same height. Timmy and the ruler actually are the same length, but because they are in relative motion, both seem shorter than the other.

7 Conclusion

We have discussed how relative motion makes clocks appear to tick more slowly and turtles appear shorter. These relationships are symmetrical; two persons in relative motion will each see the other moving more slowly, just as two persons a distance apart will each appear smaller to the other. These effects are a

[41] At this point, the reader is advised to look at both Alice's and Timmy's perspectives, understand what each line and intersection mean. Not every intersection has a label, but each intersection has a meaning. For example, the intersection of the line representing the left side of the ruler with the line representing Timmy's tail is the event, Timmy's tail passes the left side of the ruler. It may be helpful to refer back to the section on spacetime diagrams to recall how the x'-axis was constructed.

consequence of two ideas; firstly, that there is no absolute definition of stationary (statements like, "Alice is moving", are meaningless), and secondly that light is observed to travel the same speed regardless of how the observer is moving. This has been an introductory discussion of relativistic phenomena, and is by no means exhaustive. Two issues that have been omitted intentionally are the effect of relative motion on apparent *mass*[42], and how acceleration affects perception. Both of these topics are within the scope of the special theory of relativity, but sadly beyond the scope of this paper.

We have tried to emphasize that relativistic effects like time dilation and length contraction are, in a sense, illusory. This is a matter of semantics; one could always *define* size to be the same as apparent size, and thus would change as the observer moves towards or away from what is being observed. This is physically unappealing since the fundamental nature of an object should be independent of the method of observation. Discussions on relativity often, perhaps unintentionally, seem to imply the length of an object really does change based on relative motion, by using the term *rest length* to refer to the length of the object in its own frame of reference. Rest length then is independent of observational method because anyone can take their own relative motion into account and determine the rest length of an object. We claim that rest length is the more fundamental notion; what is traditionally thought of as length depends on the method of observation. We close by suggesting a change in the standard terminology, so that length refers to what would be measured in the object's own frame of reference. In doing so, we reaffirm the length of an object is independent of the distance or velocity of any observer; it is only the *apparent* size that changes corresponding to a change in the relationship between object and observer.

[42] An object appears heavier when it is moving relative to an observer.